

Heterotic strings on G_2 orbifolds

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Abstract

We study compactification of heterotic strings to three dimensions on orbifolds of G_2 holonomy. We consider the standard embedding and show that the gauge group is broken from $E_8 \times E'_8$ or $SO(32)$ to $F_4 \times E'_8$ or $SO(25)$ respectively. We also compute the spectrum of massless states and compare with the results obtained from reduction of the 10-dimensional fields. Non-standard embeddings are discussed briefly. For type II compactifications we verify that IIB and IIA have equal massless spectrum.

1 Introduction

In this note we examine compactifications of type II and heterotic strings on a class of orbifolds T^7/\mathbb{Z}_2^3 whose singularities can be resolved to get manifolds with G_2 holonomy [1–3]. One motivation to undertake this problem is that, to our knowledge, hitherto it has received little attention. Compactification of the low-energy supergravity limits on compact 7-manifolds of G_2 holonomy has been studied in [4] (see also [5]), but our main concern is to discuss the string compactification from the world sheet perspective.

It is known that string compactification on singular orbifolds is well defined as long as all twisted sectors are included and a projection on orbifold invariant states is implemented [6]. We will precisely carry out this program to systematically construct the massless states and identify their multiplicities and gauge transformation properties. We will mainly focus on heterotic strings with standard embedding of the orbifold action, but our methods can also be applied to analyze non-standard embeddings.

Various aspects of string compactification on T^7/\mathbb{Z}_2^3 orbifolds have been investigated by several authors [7–13]. Compactification of type II strings on a different class of compact G_2 manifolds was studied in [14, 15]. Closer to our endeavor is the work [16] where it was shown that in the standard embedding E_8 is broken to F_4 . We rederive this result in the orbifold construction.

In the following we will first review the basic features of the T^7/\mathbb{Z}_2^3 orbifolds, emphasizing the fixed point structure and the implications for the orbifold partition function. In section 3 we consider type II compactifications, both to obtain the full massless spectrum and to prepare the ground for the heterotic case. In section 4 we discuss compactification of the $SO(32)$ and $E_8 \times E_8'$ heterotic strings in detail, including non-standard embeddings. In order to compare results we also describe compactification on smooth 7-manifolds of G_2 holonomy. To this end for completeness in appendix 4 we construct the gravitino zero modes which also give the gaugino zero modes that determine the number of charged multiplets.

2 Joyce orbifolds

We will consider Joyce orbifolds of type T^7/Γ with automorphism group $\Gamma = \mathbb{Z}_2^3$. The torus itself is a quotient $\mathbb{R}^7/\mathbb{Z}^7$ and has coordinates (x_1, \dots, x_7) , with $x_i \equiv x_i + 1$. The generators of Γ , denoted α , β and γ , are isometries of T^7 that act on the coordinates as

$$\begin{aligned}\alpha((x_1, \dots, x_7)) &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \\ \beta((x_1, \dots, x_7)) &= (-x_1 + b_1, -x_2 + b_2, x_3, x_4, -x_5, -x_6, x_7) \\ \gamma((x_1, \dots, x_7)) &= (-x_1 + c_1, x_2, -x_3 + c_3, x_4, -x_5 + c_5, x_6, -x_7)\end{aligned}\tag{2.1}$$

where b_i and c_i are shifts equal to 0 or $\frac{1}{2}$. For concreteness we focus in two examples¹, both having $(b_1, b_2, c_1, c_5) = (0, \frac{1}{2}, \frac{1}{2}, 0)$ but distinguished by whether $c_3 = \frac{1}{2}$ in model A, or $c_3 = 0$ in model B. The shifts are appropriately chosen to ensure that after resolving the orbifold singularities the resulting manifold has G_2 holonomy [1–3].

A group element $\theta \in \Gamma$ acts as a rotation plus a translation, namely $\theta \vec{x} = \hat{\theta} \vec{x} + \vec{v}$, where $\hat{\theta} \in SO(7)$. For example, $\hat{\alpha} = \text{diag}(-1, -1, -1, -1, 1, 1, 1)$ and it is evident how it acts on vectors and tensors of $SO(7)$. It is also important to determine the action on a spinor of $SO(7)$. Acting on the 8-dimensional spinor representation, the generator corresponding to $\hat{\theta}$, denoted P_θ , must satisfy

$$P_\theta \Gamma^m P_\theta^{-1} = \hat{\theta}_n^m \Gamma^n \tag{2.2}$$

where Γ^m are Dirac matrices that fulfill $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$, $m, n = 1, \dots, 7$. It then follows that

$$P_\alpha = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \quad ; \quad P_\beta = \Gamma^1 \Gamma^2 \Gamma^5 \Gamma^6 \quad ; \quad P_\gamma = \Gamma^1 \Gamma^3 \Gamma^5 \Gamma^7 \tag{2.3}$$

These matrices commute among themselves and therefore can be diagonalized simultaneously. There is only one common eigenvector with eigenvalues $(1, 1, 1)$ under $(P_\alpha, P_\beta, P_\gamma)$. The remaining seven eigenvectors have eigenvalues that match those of $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ acting on the $SO(7)$ vector².

According to the above discussion, the spinor **8** of $SO(7)$ transforms under Γ as **1** + **7**. This is precisely the decomposition of the **8** under $G_2 \supset SO(7)$, a first hint that the

¹These are examples 3 and 4 in section 3 of [2].

²This can be checked using the 8-dimensional Γ matrices in section 8.2 of [17], multiplied by $-i$ to adjust conventions.

holonomy of the resolved T^7/Γ is G_2 . Furthermore, the group Γ preserves the 3-form

$$\phi = dx_{127} + dx_{136} + dx_{145} + dx_{235} - dx_{246} + dx_{347} + dx_{567} \quad (2.4)$$

where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The dual 4-form $^*\phi$ is also preserved. These forms are defined by the structure constants of the octonion algebra, and the subgroup of $GL(7, \mathbb{R})$ that leaves them invariant is G_2 .

2.1 The resolved orbifolds

We need to inspect in some detail the singularities of the orbifolds introduced above. In the two models distinguished by c_3 , the only elements besides the identity having fixed points are α , β and γ because other elements involve pure translations in x_1 or x_2 . In fact, the fixed sets of each element are 16 copies of T^3 . In both models, the group generated by β and γ acts freely on the fixed points of α , and similarly the fixed points of β are not left invariant by the sub-group generated by α and γ . Instead the fixed points form orbits. For instance, the 16 fixed points of α that have coordinates (x_1, x_2, x_3, x_4) with $x_i = f_i = 0, \frac{1}{2}$, span the four orbits

$$\{(0, 0, f_3, f_4) + (0, \frac{1}{2}, f_3, f_4) + (\frac{1}{2}, 0, f_3, f_4) + (\frac{1}{2}, \frac{1}{2}, f_3, f_4)\} \quad (2.5)$$

Then, the singular set of α has four components with geometry

$$T^3 \times \mathbb{C}^2/\mathbb{Z}_2 \quad (2.6)$$

where \mathbb{C}^2 has coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, and \mathbb{Z}_2 is the action $(z_1, z_2) \rightarrow (-z_1, -z_2)$. Each singularity can be resolved by replacing it with an Eguchi-Hanson space [1, 2]. The singular set of β is analogous.

The behavior of the fixed points under γ depends on the value of c_3 . In model A with $c_3 = \frac{1}{2}$, the subgroup generated by α and β acts freely on the fixed points of γ . In this case the singular set of γ also consists of four components of the form (2.6). However, in model B with $c_3 = 0$, the element $\alpha\beta$ leaves the fixed points of γ invariant. Thus, the 16 fixed points of γ organize into eight orbits, each of two elements permuted by α and β but fixed by $\alpha\beta$. They have coordinates (x_1, x_3, x_5, x_7) of the form

$$\{(\frac{1}{4}, f_3, f_5, f_7) + (\frac{3}{4}, f_3, f_5, f_7)\} \quad (2.7)$$

where $f_i = 0, \frac{1}{2}$ as before. In this orbifold the singular set of γ has eight components described by

$$\{\mathbb{T}^3 \times \mathbb{C}^2 / \mathbb{Z}_2\} / \mathbb{Z}'_2 \quad (2.8)$$

where now \mathbb{C}^2 has coordinates $z_1 = x_1 + ix_7, z_2 = x_3 + ix_5$ and \mathbb{Z}_2 acts as before. Including the \mathbb{T}^3 coordinates the action of the \mathbb{Z}'_2 generator $\alpha\beta$ is

$$\alpha\beta : (z_1, z_2, x_2, x_4, x_6) \rightarrow (z_1, -z_2, x_2 + \frac{1}{2}, -x_4, -x_6) \quad (2.9)$$

Each singular component can be repaired by using an Eguchi-Hanson (EH) space but as explained in [2, 3], there are two distinct ways of implementing the action of $\alpha\beta$. In the option to blow up the singularity the orientation of the EH space is preserved so that its fundamental 2-form ω_2 is invariant under $\alpha\beta$. If the singularity is instead deformed, the orientation is reversed and ω_2 picks up a minus sign under $\alpha\beta$.

For future purposes it is useful to review the computation of the Betti numbers of the resolved $\mathbb{T}^7 / \mathbb{Z}_2^3$ orbifolds. By Poincaré duality $b^{7-k} = b^k, k = 0, \dots, 7$. A compact connected space of G_2 holonomy has $b^0 = 1$ and $b^1 = 0$. Then, the only non-trivial Betti numbers are b^2 and b^3 . It is easy to see that the resolved orbifold indeed has $b^0 = 1$ and $b^1 = 0$, because on \mathbb{T}^7 there is one 0-form but no 1-forms invariant under Γ , and the resolution does not contribute to either b^0 or b^1 . On \mathbb{T}^7 there are no invariant 2-forms either, but there are seven Γ invariant 3-forms, namely the seven terms in ϕ in eq. (2.4). The four singular components of α are replaced by an EH space, each giving one 2-form ω_2 and three 3-forms $\omega_2 \wedge dx_i, i = 5, 6, 7$. In model A the fixed points of β and γ are repaired in the same manner. Then, the Betti numbers of the resolved orbifold, denoted, Y_A are $b^2(Y_A) = 12$ and $b^3(Y_A) = 43$.

In model B in which the singularities of γ are eight copies of (2.8), the Betti numbers b^2 and b^3 depend on how the \mathbb{Z}'_2 acts on the EH space. We will consider only the case in which all singularities are resolved in the same way. If the singularities are blown up the 2-form ω_2 of each EH space is invariant under $\alpha\beta$ and there will also be one additional invariant 3-form $\omega_2 \wedge dx_2$. In this case the Betti numbers of the resolved orbifold Y_{B1} are $b^2(Y_{B1}) = 16$ and $b^3(Y_{B1}) = 39$. Instead, when the singularities are deformed each EH adds two invariant 3-forms $\omega_2 \wedge dx_4$ and $\omega_2 \wedge dx_6$. In this situation the Betti numbers turn out to be $b^2(Y_{B2}) = 8$ and $b^3(Y_{B2}) = 47$. Notice that in both B examples, as well as in model A, the sum of b^2 and b^3 is 55.

2.2 Partition function

The structure of the singular sets can be translated into properties of the partition function of strings propagating on the T^7/Γ orbifold. Recall that when Γ is Abelian this partition function can be written as [6]

$$Z = \sum_{h \in \Gamma} Z_h = \frac{1}{|\Gamma|} \sum_{g, h \in \Gamma} Z(h, g) \quad (2.10)$$

The sum over h is over twisted sectors and the sum over g enforces the orbifold projection. Correspondingly, $Z(h, g)$ is the trace over states evaluated with boundary conditions periodic up to h in the spatial direction and up to g in the time direction of the world-sheet torus. Z_h is called the h -sector partition function.

In the T^7/Γ orbifolds under study the partition function greatly simplifies because $Z(h, g)$ vanishes when $h, g \in \Gamma$ do not have simultaneous fixed points. Moreover, for the $\Gamma = \mathbb{Z}_2^3$ that we are considering the only sectors where massless states can appear have $h = 1, \alpha, \beta, \gamma$. In the remaining twisted sectors the lowest lying states are massive because h acts as a pure translation on some coordinates in which the winding numbers must then be half-integers.

When $h = 1$, g can be any element so that in the untwisted sector partition function the sum is over all $g \in \Gamma$. On the other hand, in the α -twisted sector with $h = \alpha$, g can only be the identity or α itself because other elements act freely on the α fixed points. Therefore, in this sector

$$Z_\alpha = \frac{1}{8} [Z(\alpha, 1) + Z(\alpha, \alpha)] \quad (2.11)$$

Since α leaves 16 fixed points, we see that in Z_α the states will appear with multiplicity four, consistent with the fact that the singular set of α has four components. Notice that the orbifold projection just requires that the states in the α sector be invariant under the subgroup generated by α . In the untwisted sector the states must be invariant under the full Γ .

The contribution Z_β of the β sector is analogous, and also Z_γ in model A. In model B, the element $\alpha\beta$ leaves the fixed points of γ invariant so that

$$Z_\gamma^B = \frac{1}{8} [Z(\gamma, 1) + Z(\gamma, \gamma) + Z(\gamma, \alpha\beta) + Z(\gamma, \alpha\beta\gamma)] \quad (2.12)$$

States will appear with multiplicity 8 and they must be invariant under the subgroup generated by γ and $\alpha\beta$.

In the next sections we will study strings propagating in the Joyce orbifolds described above. We first consider type II compactification as a warm up exercise and then turn to the most interesting case of heterotic compactifications. The properties of the partition function will be essential to obtain the spectrum of massless states which will be basically determined by the Betti numbers of the resolved orbifolds.

3 Type II compactifications

To begin we quickly review the reduction of the type II supergravities on smooth manifolds of G_2 holonomy [4]. The resulting theory has four supercharges which means $\mathcal{N}=2$ supersymmetry in $d=3$. To count the number of massless multiplets it is enough to look at bosonic zero modes, taking into account that in $d=3$ a vector is dual to a scalar, and that the $\mathcal{N}=2$ scalar multiplet has a complex scalar. Reduction of the 10-dimensional NSNS fields (metric, 2-form and dilaton) to $d=3$ gives rise, on shell, to a dilaton, b^2 real scalars from the 2-form, and real metric moduli whose number is b^3 for a manifold of G_2 holonomy [18]. For type IIA the RR 1-form and 3-form reduce to one plus b^2 vectors, and b^3 real scalars. Altogether there are $(1 + b^2 + b^3)$ scalar multiplets. For type IIB, reduction of the RR even forms, with self-dual 4-form, also leads to $(1 + b^2 + b^3)$ $\mathcal{N}=2$ scalar multiplets in $d=3$. In [4] it was conjectured that IIA and IIB strings compactified on a manifold of G_2 holonomy are equivalent. We will show that type IIB and type IIA strings compactified on T^7/\mathbb{Z}_2^3 orbifolds of G_2 holonomy have equal massless spectrum as expected.

3.1 Orbifold compactification

The goal is to deduce the massless spectrum from compactification of the world sheet degrees of freedom on the orbifold. To this end we use the light cone NSR formulation and denote the left and right moving oscillators respectively by (α_r^m, ψ_s^m) and $(\tilde{\alpha}_r^m, \tilde{\psi}_s^m)$, $m = 1, \dots, 8$. The mode numbers, which depend on the sector, are all integers or half-integers in the \mathbb{Z}_2^3 orbifolds under analysis. In the untwisted sector the mass formulas for

the Neveu-Schwarz (NS) and Ramond (R) states are as usual, namely

$$\begin{aligned}\text{NS} &: M^2 = N_B + N_F - \frac{1}{2}, \\ \text{R} &: M^2 = N_B + N_F,\end{aligned}\tag{3.1}$$

where N_B and N_F are bosonic and fermionic occupation numbers. These formulas apply to both left and right movers. The massless states are the $\mathbf{8}_v \psi_{-\frac{1}{2}}^m |0\rangle$ and the $\mathbf{8}_s |s^a\rangle$ which survive the GSO projection enforced by keeping states with $e^{i\pi F} = 1$. Here $|0\rangle$ is the NS ground state, whereas the $|s^a\rangle$ Ramond states are built from the Clifford vacuum of the zero mode algebra $\{\psi_0^m, \psi_0^n\} = 2\delta^{mn}$. For right movers the results are analogous. For definiteness we focus on type IIB in which the right moving GSO projection is $e^{i\pi \tilde{F}} = 1$, both for the NS and R sectors. Type IIA will be discussed afterwards.

We next implement the orbifold projection on the untwisted states. The \mathbb{Z}_2^3 action is generated by the $SO(7)$ rotations $\hat{\alpha}, \hat{\beta}$, and $\hat{\gamma}$ under which the NS states transform as $\mathbf{8}_v = \mathbf{1} + \mathbf{7}_v$. Clearly, the singlet is $\psi_{-\frac{1}{2}}^8 |0\rangle$ whereas the $\mathbf{7}_v$ are the states $\psi_{-\frac{1}{2}}^i |0\rangle$, $i = 1, \dots, 7$. The invariant NSNS states are thus the dilaton $\tilde{\psi}_{-\frac{1}{2}}^8 |0\rangle \otimes \psi_{-\frac{1}{2}}^8 |0\rangle$, plus seven moduli $\tilde{\psi}_{-\frac{1}{2}}^i |0\rangle \otimes \psi_{-\frac{1}{2}}^i |0\rangle$. The Ramond states $|s^a\rangle$ transform as an $\mathbf{8}$ spinor of $SO(7)$, which under \mathbb{Z}_2^3 also splits as $\mathbf{1} + \mathbf{7}_v$, as we explained in the previous chapter. We will denote $|s^0\rangle$ the singlet state and $|s^i\rangle$ the remaining states transforming as $\mathbf{7}_v$. For right movers we make the analogous decomposition. The invariant RR states are therefore the axion $|\tilde{s}^0\rangle \otimes |s^0\rangle$, and seven scalar moduli $|\tilde{s}^i\rangle \otimes |s^i\rangle$.

Altogether the invariant untwisted states comprise one axiodilaton multiplet plus seven additional scalar multiplets. The states from the NSNS and RR sectors combine into complex scalars while the NSR and RNS sectors provide the fermionic partners. This result is consistent with the calculation of the spectrum by reduction of the 10-dimensional fields. We have seen that in general, besides the axiodilaton multiplet, there are $(b^2 + b^3)$ scalar multiplets. We also know that the untwisted sector corresponds to compactification on the unresolved T^7/Γ , and in T^7 there are no invariant 2-forms and 7 invariant 3-forms. Hence we indeed expect 7 additional scalar multiplets in the untwisted sector from $b_{\text{unt}}^2 = 0$ and $b_{\text{unt}}^3 = 7$.

We now analyze the twisted sectors. The twisted boundary conditions have the effect of changing the zero point energy and the mode numbers of the oscillators. Since in all

twisted sectors the $SO(7)$ rotations have four -1 eigenvalues the mass formulas become³

$$\begin{aligned} \text{NS} &: M^2 = N_B + N_F, \\ \text{R} &: M^2 = N_B + N_F. \end{aligned} \tag{3.2}$$

There are massless states because in NS, as well as in R, there are zero modes.

To be concrete we specialize to the α sector. In the following it is crucial to remember that the partition function in the α sector (2.11) just involves a projection into $\hat{\alpha}$ invariant states so that the action of $\hat{\beta}$ and $\hat{\gamma}$ is irrelevant. This enables us to bosonize the fermions ψ^i , $i = 1, 2, 3, 4$, and ψ^j , $j = 5, 6, 7, 8$, respectively into complex bosons (H_1, H_2) transforming under $\hat{\alpha}$, and (H_3, H_4) invariant under $\hat{\alpha}$. In the NS sector the zero modes are ψ_0^i , $i = 1, 2, 3, 4$, and massless states can be labelled by a $SO(4)$ spinor weight (w_1, w_2) , with $w_1, w_2 = \pm \frac{1}{2}$. The GSO projection selects $w_1 = w_2$ and the two surviving weights make up the $(\frac{1}{2}, 0)$ representation of $SO(4)$. The corresponding states, denoted $|\pm, \pm\rangle$, are invariant under $\hat{\alpha}$ that acts by multiplication by a phase $e^{i\pi(w_1 - w_2)}$. In the R sector the zero modes are ψ_0^j , $j = 5, 6, 7, 8$, and the massless states can be labelled by an $SO(4)'$ weight (w_3, w_4) , with $w_3, w_4 = \pm \frac{1}{2}$. The GSO projection picks $w_3 = w_4$ and the states, denoted $|\pm, \pm\rangle'$, are trivially invariant under $\hat{\alpha}$. For right movers the NS and R states are analogous. Requiring invariance under $\hat{\alpha}$ leaves four invariant NSNS states $|\pm, \pm\rangle \otimes |\pm, \pm\rangle$, and similarly four invariant RR states $|\pm, \pm\rangle' \otimes |\pm, \pm\rangle'$, all being real scalars. The partition function (2.11) implies that in the α sector there is an overall multiplicity of four, corresponding to the four orbits of fixed points. Therefore, in the α sector there are altogether 16 massless scalar multiplets, with fermionic partners arising in NSR and RNS sectors. This means that the contribution to the Betti numbers from the α sector is $(b_\alpha^2 + b_\alpha^3) = 16$, in agreement with the resolution of the four singular sets of α that adds one invariant 2-form and 3 invariant 3-forms per set.

We now work out the γ sector in model B, which is different because the orbifold projection requires invariance under the subgroup generated by $\hat{\gamma}$ and $\hat{\alpha}\hat{\beta}$. The massless states in the left NS sector arise from the zero modes ψ_0^k , $k = 1, 3, 5, 7$. It is convenient to bosonize (ψ^1, ψ^7) and (ψ^3, ψ^5) into $H_{(17)}$ and $H_{(35)}$, so that the massless states are

³Recall that the zero point energy of a real boson is $-\frac{1}{24} + \frac{1}{4}\delta(1 - \delta)$, with $\delta = 0$ for periodic and $\delta = \frac{1}{2}$ for antiperiodic boundary conditions. For fermions there is an overall minus sign.

labelled by weights $(w_{(17)}, w_{(35)}) = (\pm\frac{1}{2}, \pm\frac{1}{2})$. The GSO projection imposes $w_{(17)} = w_{(35)}$ and the states are invariant under $\hat{\gamma}$ that acts by multiplication by $e^{i\pi(w_{(17)}-w_{(35)})}$. Under $\hat{\alpha}\hat{\beta}$ the states are not invariant, they instead acquire a phase $e^{i\pi w_{(35)}}$. Hence, in the NSNS sector there are only two invariant states because it must be that the left and right components of the weights verify $w_{(35)} = \tilde{w}_{(35)}$. The R sector is similar. The fermions ψ^ℓ , $\ell = 2, 4, 6, 8$, are bosonized into $H_{(28)}$ and $H_{(46)}$ and the massless states are labelled by weights $(w_{(28)}, w_{(46)}) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ with GSO projection $w_{(28)} = w_{(46)}$. The states are invariant under $\hat{\gamma}$ but are multiplied by $e^{i\pi w_{(46)}}$ under $\hat{\alpha}\hat{\beta}$. Thus, in the RR sector there are also only two invariant states. Taking into account the overall multiplicity of 8 in the γ sector gives 16 massless scalar multiplets. The corresponding Betti numbers from the γ sector satisfy $(b_\gamma^2 + b_\gamma^3) = 16$, again matching the resolution that adds two invariant 3-forms or one invariant 2-form plus one invariant 3-form per each of the eight singular sets of γ .

In model A the β and γ twisted sectors are completely analogous to the α sector. In model B the γ sector is different but the net contribution is the same. Including the untwisted states the full massless spectrum in both examples consists of 56 scalar multiplets.

So far we have discussed type IIB. For type IIA we just have to change the GSO projection for the right movers in the R sector. It is easy to see that the massless spectrum remains unaltered. In the untwisted sector the spinor $\mathbf{8}_c$ states also transform as $1 + \mathbf{7}_v$ of $SO(7)$. In the α sector the GSO projection gives a $(0, \frac{1}{2})'$ of $SO(4)'$ in the right R sector, but the four RR states $|\pm, \pm\rangle' \otimes |\pm, \mp\rangle'$ are still invariant. The β sector is similar, also the γ sector in model A. In the γ sector in model B the only difference is $\tilde{w}_{(28)} = -\tilde{w}_{(46)} = \pm\frac{1}{2}$, while $w_{(28)} = w_{(46)} = \pm\frac{1}{2}$, but there are anyway 2 invariant states with $w_{(46)} = \tilde{w}_{(46)}$.

4 Heterotic compactifications

In this section we first discuss reduction of ten dimensional heterotic supergravity on a smooth manifold of G_2 holonomy. This problem has been addressed in [4] and [9] but assuming a gauge background that only leaves unbroken the maximal Abelian subgroup $U(1)^{16}$. In this paper we rather want to consider the standard embedding in which the

gauge and the spin connection are equal. This problem was already investigated in [16] where the authors give the full modular invariant partition function of the $E_8 \times E'_8$ heterotic string compactified on a T^7/\mathbb{Z}_2^3 orbifold. They also argue that E_8 is broken to F_4 , and count the massless states in the **26** representation using properties of the underlying conformal field theory. We will determine the massless spectrum using simpler standard orbifold techniques [19, 20] that can also be implemented to study other consistent embeddings.

4.1 Reduction

Upon compactification the resulting theory has two supercharges, meaning $\mathcal{N}=1$ in $d=3$. There will be neutral and charged matter in scalar multiplets that have one real scalar. There will also appear gauge multiplets with one real vector. In ten dimensions there is a gravity multiplet plus a Yang-Mills multiplet with gauge group $E_8 \times E'_8$ or $SO(32)$. Reduction of the gravity multiplet gives $(1 + b^2 + b^3)$ neutral scalar multiplets whose bosonic fields are the dilaton, the zero modes of the 2-form, and the metric moduli. The fermionic partners come from the dilatino and the components Ψ_i , $i = 1, \dots, 7$, of the gravitino. Notice in particular that by supersymmetry the Ψ_i must have $(b^2 + b^3)$ zero modes. These zero modes are constructed explicitly in the appendix.

The Yang-Mills fields give rise in $d=3$ to a gauge multiplet of the unbroken group plus charged scalar multiplets. Clearly, the bosons arise respectively from zero modes of A_μ^J and A_i^J , where J is a gauge index. To determine the resulting group and matter representations we need to specify the gauge background. In the standard embedding the gauge connection is equal to the spin connection which is a G_2 gauge field. In the $E_8 \times E'_8$ this is embedded in E_8 . The commutant of G_2 in E_8 is F_4 and to arrive at the corresponding branching it is useful to consider first the adjoint decomposition under $E_8 \supset SO(9) \times SO(7)$ given by

$$\mathbf{248} = (\mathbf{36}, \mathbf{1}) + (\mathbf{16}, \mathbf{8}) + (\mathbf{9}, \mathbf{7}) + (\mathbf{1}, \mathbf{21}) \quad (4.1)$$

Now, under $SO(7) \supset G_2$, $\mathbf{8} = \mathbf{1} + \mathbf{7}$ and $\mathbf{21} = \mathbf{7} + \mathbf{14}$, whereas under $F_4 \supset SO(9)$, $\mathbf{52} = \mathbf{36} + \mathbf{16}$ and $\mathbf{26} = \mathbf{1} + \mathbf{9} + \mathbf{16}$. Then, under $E_8 \supset F_4 \times G_2$ the adjoint branching becomes

$$\mathbf{248} = (\mathbf{52}, \mathbf{1}) + (\mathbf{26}, \mathbf{7}) + (\mathbf{1}, \mathbf{14}) \quad (4.2)$$

To look for zero modes it is actually easier to analyze the gauginos χ^J in the various representations of $F_4 \times G_2$. For G_2 singlets, i.e. $J \in (\mathbf{52}, \mathbf{1})$, the gauginos just satisfy the Dirac equation and we know that there is one solution, namely the covariantly constant spinor. This is the usual argument that explains the existence of massless gauge multiplets in the adjoint of the unbroken group. When $J \in (\mathbf{26}, \mathbf{7})$, the gauginos transform in the fundamental of G_2 , equivalently of $SO(7)$, so that they satisfy the same equation as the gravitinos Ψ_i . In the appendix we show that the Ψ_i have $(b^2 + b^3)$ zero modes, hence there will be $(b^2 + b^3)$ massless scalar multiplets transforming in the $\mathbf{26}$ of F_4 . Finally, for the gauginos in the adjoint of G_2 , $J \in (\mathbf{1}, \mathbf{14})$, we can only say that in general there will be zero modes that give massless multiplets singlets under F_4 . Clearly all these fields are neutral under the hidden E'_8 that just leads to an adjoint gauge multiplet in $d=3$.

For the $SO(32)$ heterotic string, the gauge group is broken to $SO(25)$. The adjoint decomposition under $SO(25) \times G_2$ is given by

$$\mathbf{496} = (\mathbf{300}, \mathbf{1}) + (\mathbf{25}, \mathbf{7}) + (\mathbf{1}, \mathbf{7}) + (\mathbf{1}, \mathbf{14}) \quad (4.3)$$

In this case there will be $(b^2 + b^3)$ massless scalar multiplets transforming as $\mathbf{25} + \mathbf{1}$ of $SO(25)$, plus a number of additional singlets that is not a topological invariant.

4.2 Orbifold compactification

Our purpose is to derive the massless spectrum from compactification of the world sheet degrees of freedom. For right movers we again use the light cone NSR formulation in which oscillators are denoted $(\tilde{\alpha}_r^m, \tilde{\psi}_s^m)$, $m = 1, \dots, 8$. The right massless states have been derived in the previous section, in fact, the right mass formulas are exactly as in type II, c.f. (3.1) and (3.2). We will only look into bosonic states so that we just need to consider right movers in the $\widetilde{\text{NS}}$ sector. The $\widetilde{\text{R}}$ sector leads to fermionic partners that complete full $\mathcal{N}=1$ supermultiplets in $d=3$.

The left movers include 8 real bosons and 32 real ‘gauge’ fermions λ^A . In the $SO(32)$ heterotic string all fermions belong to one set with GSO projection $e^{i\pi F} = 1$. In the $E_8 \times E'_8$ heterotic string the fermions are split into two sets λ^a and λ'^a , $a = 1, \dots, 16$, for which there are separate NS and R boundary conditions and GSO projections $e^{i\pi F} = 1$, $e^{i\pi F'} = 1$. Conventions are those of [20]. To perform the compactification it is necessary to

specify how the orbifold generators act on the gauge fermions. We will mostly focus on the standard embedding which automatically satisfies the level-matching condition required by modular invariance. For a T^7/\mathbb{Z}_2^3 heterotic orbifold the standard embedding consists of choosing a subset λ^i , $i = 1, \dots, 7$, on which the $SO(7)$ rotations $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ act in the same way as on the $\tilde{\psi}^i$. In section 4.2.3 we describe other possible actions consistent with level-matching. We work with the fermionic formulation because the form of the orbifold generators precludes combining the λ 's into complex fermions that could be bosonized. For clarity of presentation in the following we study the two heterotic theories in order.

4.2.1 Standard embedding in $SO(32)$ heterotic

Massless states can only arise in the NS sector of the left fermions in which the mass formula is given by

$$\text{NS} : M^2 = N_B + N_F + \Delta_{\text{NS}}, \quad (4.4)$$

where in the untwisted and twisted sectors $\Delta_{\text{NS}} = -1$ and $\Delta_{\text{NS}} = -\frac{1}{2}$ respectively, as can be readily checked using the results in footnote 3. In the untwisted sector the GSO projection and level matching at zero mass only allow left NS states either with one bosonic or two fermionic oscillators. Then, the untwisted $(\widetilde{\text{NS}}, \text{NS})$ massless states are

$$\tilde{\psi}_{-\frac{1}{2}}^8 |0\rangle \otimes \begin{cases} \alpha_{-1}^8 |0\rangle \\ \lambda_{-\frac{1}{2}}^I \lambda_{-\frac{1}{2}}^J |0\rangle \end{cases} \quad (4.5)$$

$$\tilde{\psi}_{-\frac{1}{2}}^i |0\rangle \otimes \begin{cases} \alpha_{-1}^i |0\rangle \\ \lambda_{-\frac{1}{2}}^i \lambda_{-\frac{1}{2}}^J |0\rangle \\ \lambda_{-\frac{1}{2}}^j \lambda_{-\frac{1}{2}}^k |0\rangle \quad ; \quad \phi_{ijk} \neq 0 \end{cases} \quad (4.6)$$

where $I, J = 8, \dots, 32$, and $i, j, k = 1, \dots, 7$. The states in (4.5) correspond to the dilaton and 300 gauge vectors (on-shell) that give the adjoint representation of $SO(25)$. The states in (4.6) are 7 metric moduli, seven scalars transforming as **25** of $SO(25)$, and 21 gauge singlets. All states are invariant under the orbifold action. In particular the gauge singlets are invariant whenever i, j, k are such that ϕ_{ijk} given in (2.4) is non-zero. We have obtained the spectrum expected from the smooth compactification. Indeed, given that $b_{\text{unt}}^2 = 0$ and $b_{\text{unt}}^3 = 7$, there must be 7 metric moduli and seven scalars transforming as $(\mathbf{25} + \mathbf{1})$ of $SO(25)$. There are also 14 additional gauge singlets.

In the twisted sectors the gauge fermions λ^i behave analogous to the ψ^i so that we can use the results of section (3.1). We consider first the α sector in which the left vacuum is a $SO(4)$ spinor because there are zero modes λ_0^m , $m = 1, 2, 3, 4$. Taking into account the GSO and orbifold projections the massless states are found to be

$$|\pm, \pm\rangle \otimes \begin{cases} \lambda_{-\frac{1}{2}}^I |\pm, \pm\rangle \\ \lambda_{-\frac{1}{2}}^\ell |\pm, \pm\rangle \\ \alpha_{-\frac{1}{2}}^m |\pm, \mp\rangle \end{cases} \quad (4.7)$$

where $\ell = 5, 6, 7$. In the α sector there is an overall multiplicity of 4 due to the number of fixed orbits. Therefore, altogether matter comprises 16 scalars in the **25** of $SO(25)$ and $16 \cdot 3$ gauge singlets. There are also $16 \cdot 4$ singlets which presumably entail metric moduli and blowing-up modes as in Calabi-Yau orbifolds with standard embedding [20,21]. However, although the states include bosonic oscillators acting on the vacuum as in the Calabi-Yau orbifolds, they are all gauge singlets. Hence, after resolving singularities the gauge group remains $SO(25)$, in agreement with the smooth compactification carried out before.

For model B we need to elucidate the γ sector in which there are zero modes λ_0^n , $n = 1, 3, 5, 7$. The vacuum is given by $SO(4)$ spinors also denoted $|\pm, \pm\rangle$ and $|\pm, \mp\rangle$. Now we have to impose invariance under $\hat{\gamma}$ and $\hat{\alpha}\hat{\beta}$. The resulting massless states are

$$|+, +\rangle \otimes \begin{cases} \lambda_{-\frac{1}{2}}^I |+, +\rangle \\ \lambda_{-\frac{1}{2}}^2 |+, +\rangle \quad ; \quad \lambda_{-\frac{1}{2}}^{4,6} |-, -\rangle \\ \alpha_{-\frac{1}{2}}^{1,7} |-, +\rangle \quad ; \quad \alpha_{-\frac{1}{2}}^{3,5} |+, -\rangle \end{cases} \quad (4.8)$$

$$|-, -\rangle \otimes \begin{cases} \lambda_{-\frac{1}{2}}^I |-, -\rangle \\ \lambda_{-\frac{1}{2}}^2 |-, -\rangle \quad ; \quad \lambda_{-\frac{1}{2}}^{4,6} |+, +\rangle \\ \alpha_{-\frac{1}{2}}^{1,7} |+, -\rangle \quad ; \quad \alpha_{-\frac{1}{2}}^{3,5} |-, +\rangle \end{cases} \quad (4.9)$$

Clearly the spectrum has a different structure compared to that in the α sector. However, since the overall multiplicity due to fixed sets is now 8, in total there are again 16 scalars in the **25** of $SO(25)$, $16 \cdot 3$ gauge singlets and $16 \cdot 4$ metric moduli plus blowing-up modes.

The massless spectrum in the twisted sectors also agrees with the smooth compactification. It happens that $(b_{\text{twi}}^2 + b_{\text{twi}}^3) = 16$ and in fact in each sector we have found 16

scalars transforming as $(\mathbf{25} + \mathbf{1})$ of $SO(25)$ plus 32 additional gauge singlets. Concerning the remaining $16 \cdot 4$ singlets per sector, we expect that $16 \cdot 3$ become massive upon blowing-up while 16 remain massless as metric moduli.

In conclusion, the gauge group is broken to $SO(25)$ and there are $b^2 + b^3 = 55$ multiplets transforming as $(\mathbf{25} + \mathbf{1})$. There are also 110 gauge bundle moduli, 14 from the untwisted sector and 32 from each twisted sector. The remaining states are 55 moduli multiplets plus 144 additional singlets that presumably become massive after resolving the singularities.

4.2.2 Standard embedding in $E_8 \times E'_8$ heterotic

There is now a (NS, NS') sector for the left fermions in which the mass formula is just (4.4) replacing $N_F \rightarrow N_F + N'_F$. It is easy to see that in the untwisted sector the massless states are as in (4.5) and (4.6) but with $I, J = 8, \dots, 16$. Thus, there will be 36 gauge vectors that furnish the adjoint of $SO(9)$ and 7 scalars that transform as $\mathbf{9}$. There are new states $\tilde{\psi}_{-\frac{1}{2}}^8 |0\rangle \otimes \lambda_{-\frac{1}{2}}^a \lambda_{-\frac{1}{2}}^b |0\rangle$ that are vectors in the adjoint of $SO(16)'$. Recall that in the standard embedding the fermions λ^a are totally inert under the orbifold action.

The main new feature in the $E_8 \times E'_8$ heterotic is the existence of massless states in mixed sectors of the left fermions in which the mass formula turns out to be

$$(\text{NS}, \text{R}'), (\text{R}, \text{NS}') : M^2 = N_B + N_F + N'_F \quad (4.10)$$

for both untwisted and twisted sectors. (NS, R') only leads to gauge vectors that transform in the $\mathbf{128}$ of $SO(16)'$ and complete the adjoint of E'_8 . In the untwisted (R, NS') sector there are zero modes λ_0^a and the vacuum is a $SO(16)$ spinor. The GSO projection selects the $\mathbf{128}$ that under $SO(7) \times SO(9)$ transforms as $(\mathbf{8}, \mathbf{16})$. We already know that the $\mathbf{8}$ spinor of $SO(7)$ transforms as $\mathbf{1} + \mathbf{7}_v$ under the orbifold \mathbb{Z}_2^3 . Therefore, massless states can be denoted $|s^0, S\rangle$ and $|s^i, S\rangle$, $i = 1, \dots, 7$, where S stands for the $\mathbf{16}$ spinor of $SO(9)$. The orbifold invariant states in $(\widetilde{\text{NS}}, \text{R}, \text{NS}')$ are then

$$\tilde{\psi}_{-\frac{1}{2}}^8 |0\rangle \otimes |s^0, S\rangle \quad ; \quad \tilde{\psi}_{-\frac{1}{2}}^i |0\rangle \otimes |s^i, S\rangle \quad (4.11)$$

Combining with states from $(\widetilde{\text{NS}}, \text{NS}, \text{NS}')$ gives 52 gauge vectors that provide the adjoint of F_4 . In fact, under $F_4 \supset SO(9)$, $\mathbf{52} = \mathbf{36} + \mathbf{16}$. We find also 7 scalars transforming in the fundamental $\mathbf{26}$ of F_4 decomposed as $(\mathbf{1} + \mathbf{9} + \mathbf{16})$ under $SO(9)$. There remain 14 extra gauge singlets.

Consider now the α twisted sector. There are massless solutions of (4.10) on account of the zero modes $\lambda_0^{5,\dots,16}$. The vacuum is a $SO(12)$ spinor which is α invariant. The GSO projection selects the **32** that transforms as $(\mathbf{2}, \mathbf{16})$ under $SO(3) \times SO(9)$. We label the states as $|\sigma, S\rangle$ with σ the **2** spinor of $SO(3)$. The invariant states in $(\widetilde{\text{NS}}, \text{R}, \text{NS}')$ are simply

$$|\pm, \pm\rangle \otimes |\sigma, S\rangle \quad (4.12)$$

which correspond to 4 scalars transforming as **16** of $SO(9)$. The $(\widetilde{\text{NS}}, \text{NS}, \text{NS}')$ massless states are read from (4.7). We see that among them there are 4 scalars transforming as $\mathbf{1} + \mathbf{9}$ of $SO(9)$. Including the multiplicity 4 due to the fixed sets we conclude that there are 16 scalars in a full **26** of F_4 . There remain $16 \cdot 2$ gauge singlets.

Presently we analyze the γ sector in the B model. In (R, NS') there are zero modes $\lambda_0^{2,4,6}$ and $\lambda_0^{8,\dots,16}$. After the GSO projection the vacuum is again a $SO(12)$ spinor **32** that decomposes as $(\mathbf{2}, \mathbf{16})$ of $SO(3) \times SO(9)$. To perform the orbifold projection we need to determine how the spinor **2** of $SO(3)$ transforms under $\hat{\alpha}\hat{\beta}$. The $SO(3)$ restricts to the coordinates (x_2, x_4, x_6) on which $\hat{\alpha}\hat{\beta}$ acts as $\text{diag}(1, -1, -1)$. In the spinor representation the action is $P_{\alpha\beta} = \gamma^2\gamma^3$ where the three 2-dimensional Dirac matrices satisfy $\{\gamma^i, \gamma^j\} = \delta^{ij}$. Clearly $P_{\alpha\beta}$ has eigenvalues $\pm i$, the corresponding eigenstates are called σ_{\pm} . Thus, the γ invariant states in $(\widetilde{\text{NS}}, \text{R}, \text{NS}')$ are

$$|+, +\rangle \otimes |\sigma_+, S\rangle \quad ; \quad |-, -\rangle \otimes |\sigma_-, S\rangle \quad (4.13)$$

where S again stands for the **16** spinor of $SO(9)$. Combining with states in $(\widetilde{\text{NS}}, \text{NS}, \text{NS}')$ given in (4.8) and (4.9), yields 2 scalars transforming in the **26** of F_4 . Since the fixed set multiplicity is 8 in the end the overall spectrum is the same as in the α sector studied before.

The final outcome is that the massless orbifold spectrum conforms with reduction on a smooth manifold. E_8 is broken to F_4 and there are 55 multiplets transforming in the **26**. The counting of additional singlets is exactly as in the $SO(32)$ heterotic. Similar results have been obtained in [16].

4.2.3 Non-standard embeddings

The orbifold action on the right fermions $\tilde{\psi}^i$ is given by the $SO(7)$ rotations $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ defined in (2.1). The embedding in the left fermions λ^A is realized by gauge twists of

order two denoted (A, B, C) . These twists can be taken to be diagonal and are specified by strings of (-1) 's and 1 's. For instance, in the standard embedding

$$\begin{aligned} A_0 &= (-1^4, 1^{12}; 1^{16}) \\ B_0 &= (-1^2, 1^2, -1^2, 1^{10}; 1^{16}) \\ C_0 &= (-1, 1, -1, 1, -1, 1, -1, 1^9; 1^{16}) \end{aligned} \tag{4.14}$$

where $(\pm 1)^n$ stands for (± 1) repeated n times. The separation into two groups of 16 eigenvalues is meant to apply only to the $E_8 \times E'_8$ heterotic string.

To determine the level-matching constraints we take a generic twist with t eigenvalues equal to (-1) . The zero point energy for all left fermions in NS sector is easily found to be $\Delta_{\text{NS}} = \frac{t-12}{16}$, while $\Delta_{\text{R}} = \frac{20-t}{16}$ in the R sector. On the other hand, for the right movers, $\tilde{\Delta}_{\text{NS}} = \tilde{\Delta}_{\text{R}} = 0$. Therefore, level-matching requires that t take values 4, 12, 20, 28, so that Δ_{NS} is a multiple of $\frac{1}{2}$, as the occupation numbers in the twisted sectors. In the $E_8 \times E'_8$ heterotic string the (-1) eigenvalues can be distributed between the two sets of fermions but the number on each set has to be a multiple of 4. This last condition guarantees that the zero point energy in mixed (R, NS'), (NS', R) sectors is a multiple of $\frac{1}{2}$ as well. It can also be understood in the bosonic formulation in which a \mathbb{Z}_2 twist must correspond to a shift vector V such that $2V$ belongs to the $E_8 \times E'_8$ root lattice.

For a single twist some possibilities are equivalent, e.g. $t = 12$ and $t = 20$ in $SO(32)$, but all have to be taken into account to obtain the allowed triplets (A, B, C) . Notice that the products AB , BC and AC must also satisfy the condition on the number of negative eigenvalues. We will not attempt to classify all allowed embeddings. We will just give some simple examples to show how the standard orbifold techniques can be applied to derive the spectrum.

The first example in the $SO(32)$ heterotic has twists (A_1, B_0, C_0) , with

$$A_1 = (-1^4, 1^3, (-1)^8, 1^{17}) \tag{4.15}$$

and B_0, C_0 as in (4.14). We will describe the spectrum briefly, concentrating in the differences with the standard embedding. The gauge group turns out to be $SO(9) \times SO(17)$. The less evident $SO(9)$ gauge vectors arise from untwisted states $\tilde{\psi}_{-\frac{1}{2}}^8 |0\rangle \otimes \lambda_{-\frac{1}{2}}^r \lambda_{-\frac{1}{2}}^s |0\rangle$, $r, s = 4, 8, \dots, 15$, which happen to be invariant. The untwisted charged

matter consists of $[6(\mathbf{9}, \mathbf{1}) + 6(\mathbf{1}, \mathbf{17}) + (\mathbf{9}, \mathbf{17})]$. For instance, the mixed states are $\tilde{\psi}_{-\frac{1}{2}}^4 |0\rangle \otimes \lambda_{-\frac{1}{2}}^r \lambda_{-\frac{1}{2}}^L |0\rangle$, $L = 12, \dots, 32$. In the α twisted sector there are massless states in the NS sector of the left fermions but now the zero point energy vanishes. The zero modes are $\lambda_0^{1,2,3}, \lambda_0^r$. Then, altogether the massless charged states are $16(\mathbf{16}, \mathbf{1})$, where $\mathbf{16}$ is the $SO(9)$ spinor. The charged states in the β and γ sectors are basically as in the standard embedding. In each case we find $16[(\mathbf{9}, \mathbf{1}) + (\mathbf{1}, \mathbf{17})]$.

A non-standard embedding in the $E_8 \times E'_8$ heterotic is obtained with twists (\tilde{A}_1, B_0, C_0) , where now

$$\tilde{A}_1 = (-1^4, 1^3, (-1)^4, 1^5; (-1)^4, 1^{12}) \quad (4.16)$$

The eight additional (-1) eigenvalues are distributed between the two factors to avoid reobtaining the $F_4 \times E'_8$ model. The gauge group is found to be $SO(9) \times E'_7 \times SU(2)'$. The charged matter spectrum can be determined as in the previous examples. For instance, in the α sector there are $16(\mathbf{16}, \mathbf{1}', \mathbf{2}')$.

5 Final Comments

The aim of this paper was to study the compactification of heterotic strings on T^7/\mathbb{Z}_2^3 orbifolds. Using systematic orbifold techniques we were able to find the invariant massless states in the untwisted and twisted sectors. In the standard embedding the results match those obtained from reduction of the 10-dimensional theory on a smooth manifold of G_2 holonomy. Concretely, the gauge group $SO(32)$ or $E_8 \times E'_8$ is broken to $SO(25)$ or $F_4 \times E'_8$ respectively. Furthermore, there are $(b^2 + b^3)$ multiplets transforming in the fundamental of F_4 or $SO(25)$, as well as an equal number of moduli multiplets. The Betti numbers are those of the resolved orbifold. There are additional gauge bundle moduli whose number is also determined in the orbifold construction. Our methods naturally apply to investigate non-standard embeddings and we have provided some examples.

We have also shown that type IIB and type IIA strings compactified on T^7/\mathbb{Z}_2^3 orbifolds of G_2 holonomy have equal massless spectrum consisting of $(1 + b^2 + b^3)$ $\mathcal{N}=2$ scalar multiplets in $d=3$.

Our main motivation was to study standard and non-standard heterotic compactifications to uncover the unbroken gauge symmetries. The allowed Higgsing patterns in

the resulting $d=3$ $\mathcal{N}=1$ supersymmetric gauge theories could be determined. It would be of interest to compare with M-theory compactifications on 8-dimensional manifolds of Spin(7) holonomy to understand the enhancing to non-simply laced groups.

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A Gravitino zero modes

By supersymmetry the gravitino Ψ_p , $p = 1, \dots, 7$, must have $(b^2 + b^3)$ zero modes. This result can be shown using general properties of a manifold Y of G_2 holonomy. On Y there is a covariantly constant spinor η and in consequence there exist a covariantly constant 3-form φ and a 4-form $\Phi = *\varphi$ given by [18]

$$\varphi_{mnp} = i\bar{\eta}\Gamma_{mnp}\eta \quad ; \quad \Phi_{mnpq} = -\bar{\eta}\Gamma_{mnpq}\eta \quad (\text{A.1})$$

We use the conventions of [22].

Zero modes of Ψ_p satisfy the Rarita-Schwinger equation

$$\Gamma^{mnp}D_n\Psi_p = 0 \quad (\text{A.2})$$

In Y we can construct the solutions using the covariantly constant spinor and the harmonic forms, as it is done in Calabi-Yau compactifications [23]. For example, a natural Ansatz consists of

$$\Psi_p^{(0)} = \varphi_{prs}\Gamma^{rs}\eta \quad (\text{A.3})$$

It immediately follows that $\Psi_p^{(0)}$ satisfies (A.2) because φ is covariantly constant. This mode has $\Gamma^p\Psi_p^{(0)} \neq 0$ and is related by supersymmetry to the volume modulus of the metric.

The remaining harmonic 3-forms, $a_{prs}^{(\tau)}$, $\tau = 1, \dots, b^3 - 1$, give rise to

$$\Psi_p^{(\tau)} = a_{prs}^{(\tau)}\Gamma^{rs}\eta \quad (\text{A.4})$$

Using identities of antisymmetrized products of Γ matrices [24] and the fact that the $a^{(\tau)}$ are closed and co-closed we find that $\Gamma^{mnp} D_n \Psi_p^{(\tau)} = \frac{2}{3} g^{mr} D_r (\Gamma^{nps} a_{nps}^{(\tau)} \eta)$. Furthermore, $\Gamma^p \Psi_p^{(\tau)} = \Gamma^{nps} a_{nps}^{(\tau)} \eta$. Thus, the Ansatz (A.4) is traceless and fulfills the Rarita-Schwinger equation provided that

$$a_{nps}^{(\tau)} \Gamma^{nps} \eta = 0 \quad (\text{A.5})$$

To prove that the $a^{(\tau)}$ satisfy this condition we have to rely on further properties of G_2 manifolds.

It is known that the harmonic forms on Y split according to the branching of $SO(7)$ representations into those of G_2 [3]. The Betti numbers reflect this decomposition. For instance, since a 3-form transforms in the **35** of $SO(7)$ it follows that $b^3 = b_1^3 + b_7^3 + b_{27}^3$, where the subscripts indicate the G_2 representation. Moreover, on Y $b_7^3 = 0$ and $b_1^3 = 1$ corresponds to the invariant 3-form φ . The remaining 3-forms $a^{(\tau)}$ are in the **27** and are characterized by $a^{(\tau)} \wedge \varphi = 0$ and $a^{(\tau)} \wedge \Phi = 0$ [25]. These two conditions in turn imply

$$a_{mnp}^{(\tau)} \Phi^{\ell mnp} = 0 \quad ; \quad a_{mnp}^{(\tau)} \varphi^{mnp} = 0 \quad (\text{A.6})$$

Now, given (A.1), it can be shown that [22]

$$\Gamma^{mnp} \eta = -i \varphi^{mnp} \eta + \Phi^{mnp\ell} \Gamma_\ell \eta \quad (\text{A.7})$$

Contracting with $a_{mnp}^{(\tau)}$ and substituting (A.6) yields the desired result (A.5).

Harmonic 2-forms, $a_{pq}^{(v)}$, $v = 1, \dots, b^2$, give further zero modes

$$\Psi_p^{(v)} = a_{pq}^{(v)} \Gamma^q \eta \quad (\text{A.8})$$

We now obtain $\Gamma^{mnp} D_n \Psi_p^{(v)} = -\frac{1}{2} g^{mr} D_r (\Gamma^{np} a_{np}^{(v)} \eta)$ and $\Gamma^p \Psi_p^{(v)} = \Gamma^{np} a_{np}^{(v)} \eta$. Then, the condition

$$a_{np}^{(v)} \Gamma^{np} \eta = 0 \quad (\text{A.9})$$

guarantees that the $\Psi_p^{(v)}$ satisfy $\Gamma^p \Psi_p^{(v)} = 0$, and the Rarita-Schwinger equation. The number of harmonic 2-forms splits as $b^2 = b_7^2 + b_{14}^2$, and $b_7^2 = 0$ [3]. Thus, the $a^{(v)}$ are in the **14** and are characterized by $a^{(v)} \wedge \varphi = - * a^{(v)}$ [25]. Taking dual gives $a_{mn}^{(v)} = -\frac{1}{2} a^{(v) pq} \Phi_{mnpq}$. Contracting with $\varphi^{mn\ell}$ we obtain

$$a_{mn}^{(v)} \varphi^{mn\ell} = 0 \quad (\text{A.10})$$

by virtue of the identity $\Phi_{mnpq}\varphi^{mn\ell} = 4\varphi_{pq}^{\ell}$ (see e.g. appendix B in [22]). The result (A.9) finally follows from $\Gamma^{np}\eta = -i\varphi^{np\ell}\Gamma_{\ell}\eta$.

In conclusion, the $(b^3 + b^2)$ gravitino zero modes are given by (A.3), (A.4), and (A.8).

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